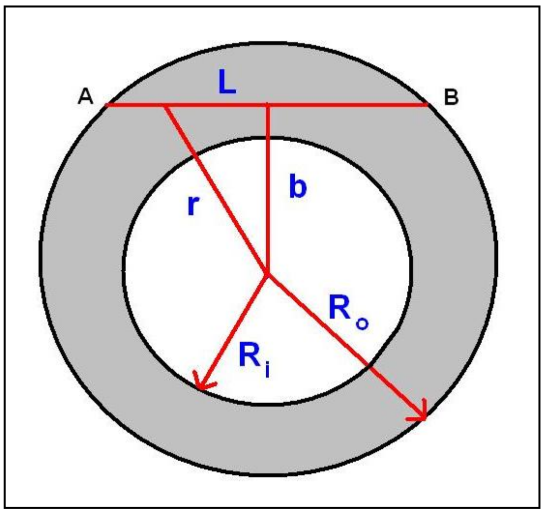
Modeling a Planetary Nebula

Planetary nebula are the outer atmospheres of dying stars ejected into space. Astronomers model these nebulae to learn about the total mass they contain, and the details of how they were ejected. The image is of a rare, spherical-shell planetary nebula, Abell 38, photographed by astronomer George Jacoby (WIYN Observatory) and his colleagues using the giant, 4-meter Mayall Telescope at Kitt Peak, Arizona. Abell-38 is located 7,000 light years away in the constellation Hercules. The nebula is 5 light years in diameter and 1/3 light year thick. For other spectacular nebula images, visit the Hubble Space Telescope archive at
http://hubblesite.org/newscenter/archive/releases/nebula



Statement of the Problem:

We want to calculate the intensity of the nebula (shaded shell) at different radii from its center (b) along a series of chords through the nebula (AB). The intensity, I(b) will be proportional to the density of gas within the nebula, which we define as D(r).

The shell is spherically-symmetric, as is D(r), so there are obvious symmetries in the geometry of the problem.

Because D(r) varies along the chord AB, we have to sum-up the contribution to I(b) from each spot along AB.

Problem 1 - Using the Pythagorean Theorem, define L in terms of b and r.

Problem 2 - Calculate the differential, dL in terms of r and b, assuming b is a constant.

Problem 3 - Construct the differential dI(b,r) = D(r)dL explicitly in terms of r and b.

Problem 4 - Integrate dI(b,r) to get I(b) with the assumption that D(r) = 0 from r = 0 to r = R_i, and is constant throughout the shell from r=R_i to r = R_0, and that D(r) = D_0.

Problem 5 - Assuming that all linear units are in light years, plot the 1-d function I(b) for R_i = 2.2 and R_0 = 2.5, and from b=0 to b=5.0, and compare it with Abell-38. Does Abell-38 seem to follow a constant-density shell model?

Answer Key

Problem 1 - $L^2 = r^2 - b^2$

Problem 2 - $2L\,dL = 2r\,dr$ so $dL = r\,dr / L$, and by substitution for $L = (r^2 - b^2)^{1/2}$, you get

$$dL = r\,dr / (r^2 - b^2)^{1/2}$$

Problem 3 - Construct the differential $dI(b,r) = D(r)dL$ explicitly in terms of r and b.

$$dI = D(r)\, r / (r^2 - b^2)^{1/2}\, dr$$

Problem 4 - Integrate $dI(b,r)$ to get $I(b)$ with the assumption that $D(r) = 0$ from $r = 0$ to $r = R_i$, and is constant throughout the shell from $r=R_i$ to $r = R_0$, and that $D(r) = D_0$.

$$I(b) = \int_{R_i}^{R_o} \frac{D_o\, r}{\sqrt{r^2 - b^2}}\, dr$$

D_0 is a constant, so it can be factored out from the integrand. Use the substitution $U = r^2 - b^2$, so that $dU = 2r\,dr$, and the integrand becomes $(1/2)\, D_0\, dU/U^{1/2}$. The integral can easily be solved to get $D_0\, (r^2 - b^2)^{1/2} + C$. Evaluating this 'front half integral' for the stated limits, and remembering that there are two halves to the shell, leads to the function:

$$I(b) = 2\, D_0\, (R_0^2 - b^2)^{1/2} - 2D_0\, (R_i^2 - b^2)^{1/2}.$$

Problem 5 - There are actually two parts to evaluating this function, making it a 'piecewise' function. The first part is for b <= 2.2 ly, for which you get:

$$I(b) = 2D_0\, (6.25 - b^2)^{1/2} - 2D_0\, (4.84 - b^2)^{1/2}$$

The second part is the solution for b > 2.2 ly for which you will get

$$I(b) = 2D_0\, (6.25 - b^2)^{1/2}$$

This function, plotted below, does seem to show an increase in brightness in the nebula between R=2.0 and 2.5 light years which matches the picture of the nebula Abell-38, so the nebula may be a hollow, thin spherical shell with a uniform density of gas!